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COMMENT

Remarks on negative energy states in supersymmetric quantum mechanics

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Abstract. We analyse the role of singular superpotentials in supersymmetric quantum mechanics. In particular, we investigate the existence of negative energy states in supersymmetry and analyse the behaviour of superpartners in such cases.

Supersymmetric quantum mechanics (SUSYQM) (Witten 1981) is the simplest system which allows Bose-Fermi symmetry. It can be regarded as a field theory in $(0+1)$ dimensions and various ideas can be tested within this framework in a simple manner. For example, it has been shown (Cooper and Freedman 1983) that SUSY breaking can be obtained if the superpotential is chosen suitably. However, apart from a few exceptions (Jevicki and Rodrigues 1984) most of the papers deal with non-singular superpotentials and the role of singular superpotentials in SUSYQM have not been investigated in detail. In this comment we shall treat SUSYQM models with singular superpotentials and discuss the occurrence of (normalisable) negative energy states in such systems. The models to be used are a family of double-well potentials and their SUSY partners.

A SUSYQM model in one dimension is specified by a pair of Hamiltonians (Cooper and Freedman 1983)

$$H=\begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}=\{Q^+, Q\} \tag{1}$$

$$H_\pm=-\frac{d^2}{dx^2}+V_\pm(x) \tag{2}$$

$$V_\pm(x)=W^2(x)\pm W'(x) \tag{3}$$

$$Q=(p-iW)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{4}$$

$$Q^+=(p+iW)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{5}$$

In the above Q and Q^+ are called the supercharges and $W(x)$ is called the superpotential. Here an important observation is that the zero-energy states corresponding to $H_\pm$ are given by

$$\varphi^0_\pm(x)=C\exp\left(\pm\int^x W(t)\,dt\right) \tag{6}$$

where C is a normalisation constant.

In the present case we take $W(x)$ to be (Roy and Roychoudhury 1987)

$$W(x)=x^3-\frac{c}{x}-\sum_{i=0}^{N}\frac{2g_ix}{(1+g_ix^2)} \qquad g_0=0. \tag{7}$$

This is the general form of the superpotential for the double-well potentials of the type

$$V_-(x)=x^6-(2n+1)x^2 \qquad (n=0,1,2,\ldots).$$

Let us first analyse the case $c=1$, $N=0$. In this case we get from (3)

$$V_-(x)=x^6-5x^2 \tag{8}$$

$$V_+(x)=x^6+x^2+2/x^2. \tag{9}$$

Note that $W(x)$ and $V_+(x)$ have a singularity at the origin. Also from (6) we have

$$\varphi_+^0(x)=C_1x^{-1}\exp(\tfrac{1}{4}x^4) \tag{10}$$

$$\varphi_-^0(x)=\left(\frac{2^{1/4}}{\Gamma(3/4)}\right)^{1/2}x\exp(-\tfrac{1}{4}x^4). \tag{11}$$

It is clear that while $\varphi_-^0(x)$ is normalisable, $\varphi_+^0(x)$ is not. The form of $\varphi_-^0(x)$ (equation (11)) suggests that the H_- sector has negative energy states. This can also be seen from the following arguments. Let

$$f_0=\exp(-\tfrac{1}{4}x^4). \tag{12}$$

This is the ground state of H_-^R given by

$$H_-^R=H_-+2x^2=(-\partial+x^3)(\partial+x^3) \tag{13}$$

where

$$H_-=-\frac{d^2}{dx^2}+x^6-5x^2 \tag{14}$$

then

$$(f_0,H_-f_0)=-2(f_0,x^2f_0)<0. \tag{15}$$

Hence H_- is a negative operator ($-d^2/dx^2+x^6-\gamma x^2$ is always a negative operator for $\gamma>3$). Now from (1),

$$H=\begin{pmatrix}H_- & 0\\ 0 & H_+\end{pmatrix}=K^2 \tag{16}$$

(say) where

$$K=\begin{pmatrix}0 & -\partial+w\\ \partial+w & 0\end{pmatrix} \tag{17}$$

and

$$W(x)=x^3-1/x. \tag{18}$$

Since H_- is a negative operator, K^2 is not positive and K cannot be self-adjoint. But since H_- is obviously self-adjoint H_+ cannot be self-adjoint. Also H_+ is unbounded. This follows from the following arguments. Let $\psi_0(x)=e^{-x^2/2}$ (ground-state harmonic oscillator wavefunction); then $(\psi_0,H_+\psi_0)$ involves an integral proportional to

$$\int_0^\infty\frac{e^{-x^2}}{x^2}\,dx$$

which does not exist because of the singularity at $x=0$. Hence $H_+ \not\subset B(H)$† and therefore though H_+ is Hermitian it is not self-adjoint and it may have complex eigenvalues (see Dunford and Schwarz 1965). In fact, the above arguments hold for a more general superpotential than x^3-1/x. Consider a superpotential $W^R(x)$ which is positive and such that $W^R(x)/x$ is integrable near the origin. Then consider the Hamiltonian $H^R = K^{R^2}$ where

$$K^R = \begin{pmatrix} 0 & -\partial + W^R \\ \partial + W^R & 0 \end{pmatrix}.$$

Now consider the superpotential W given by $W = W^R - 1/x$ and the corresponding Hamiltonian H_- given by

$$H = \begin{pmatrix} 0 & -\partial + W \\ \partial + W & 0 \end{pmatrix}.$$

Clearly

$$H_- = H^R - 2W^R/x$$

which is self-adjoint but

$$(f_0, H_- f_0) = -2\int f_0^2 \frac{W^R}{x}\,\mathrm{d}x$$

is definitely negative, f_0 being the ground state corresponding to H_-^R. But K^2 is not positive and H_+ will not be self-adjoint. As has been shown by Roy and Roychoudhury (1987) the superpotential $2gx/(1+gx^2)$ also has this property because it drops out in H_- but appears in H_+ and if $g<0$, the singularity appears in H_+ which is absent in H_-. Some numerical results where the superpotential contains a term like $2gx/(1+gx^2)$ are given below. (Though the exact numerical value of the negative energy state is not required for the above arguments the numerical values of energy for the ground state and the first few excited states for the potential $V(x)=x^6-5x^2$ and $V(x)=x^6-7x^2-2\sqrt{2}$ are presented here (table 1) for future comparison.) It may be pointed out that the ground state for the potential $x^6-7x^2-2\sqrt{2}$ can be calculated exactly and is found to be $\psi_0=(1+2x^2)\,\mathrm{e}^{-x^4/4}$ with eigenvalue $-4\sqrt{2}$ while its zero-energy state is $\psi_0=(1-2x^2)\,\mathrm{e}^{-x^4/4}$.

Table 1. Energy value corresponding to $W=x^3-1/x$ and $W=x^3+2\sqrt{2}x/(1-\sqrt{2}x^2)$.

n	$V_-(x)=x^6-5x^2$†	$V_-(x)=x^6-7x^2-2\sqrt{2}$
0	−1.153 54	−5.6568
1	0.0	−5.0935
2	5.047 803	0
3	9.455 53	4.3696
4	15.539 11	10.1807
5	22.503 93	16.8930

† See also Boya *et al* (1987).

† Here $B(H)$ denotes the Banach algebra of all bounded linear operators T on a Hilbert space H.

Usually the quantity which is of much importance in double-well potentials is the difference between the two lowest eigenvalues given by $t = E_1 - E_0$ as it corresponds to the tunnelling route through the double-well barrier. As the coefficient of x^2 increases in magnitude the quantity t becomes very small and is difficult to calculate numerically. Some authors (Keung *et al* 1988, Bernstein and Brown 1984) claim that supersymmetry facilitates the evaluation of t in these cases. They actually calculate the ground state of the superpartner Hamiltonian H_+ and assume that this is the same as the first excited state of H_-, i.e. the degeneracy holds. But as the potential barrier increases, H_- can have negative eigenvalues and as we have shown in this comment, $H_+(x)$ ceases to become self-adjoint and the degeneracy argument breaks down (also see Deift 1978).

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